

The reduction of the problem of maximization of the fraction of two functionals

I.P. Smirnov

Institute of Applied Physics RAS,
46 Ul'yanova Street, Nizhny Novgorod, Russia

Abstract

We propose an algorithm for reduction of the problem of maximization of fraction of two functionals to the equivalent procedure including maximization of difference between the functionals and the solution of an equation of scalar unknown. For illustration of the algorithm we solve some problems of the described type.

Key words: extremal problem, iteration scheme

1 Introduction

Theorem 1. *Let $W(x) > 0$, $W_0(x)$ be continuous functionals on a compact set K of a metric space, β be a number,*

$$J(x) \equiv \frac{W_0(x)}{W(x)},$$

$$J_\beta(x) \equiv W_0(x) - \beta W(x).$$

Consider the following extremal problems

$$J \rightarrow \max_K, \tag{1}$$

and

$$J_\beta \rightarrow \max_K. \tag{2}$$

Let x_{\max} be a solution of the problem (1), x_β a solution of (2),

$$\beta_{\max} \equiv J(x_{\max}), \quad j(\beta) \equiv J_\beta(x_\beta).$$

Then

$$\begin{cases} j(\beta) < 0, & \beta > \beta_{\max}, \\ j(\beta) > 0, & \beta < \beta_{\max}, \\ j(\beta) = 0, & \beta = \beta_{\max}, \end{cases} \quad (3)$$

so β_{\max} is the only solution of the equation

$$j(\beta) = 0. \quad (4)$$

Functionals $J(x)$ and $J_{\beta_{\max}}(x)$ take their maxima at the same point $x_{\max} = x_{\beta_{\max}}$. The function $\beta \rightarrow x_\beta$ is continuous in every compact segment $[\beta_1, \beta_2]$.

Prove. Under the hypotheses of the theorem, the solutions of the extrema problems x_{\max} , x_β exist (may be not unique). If $\beta > \beta_{\max}$, then $J_\beta(x) = W(x)(J(x) - \beta) < 0$ for all x ; so $j(\beta) = J_\beta(x_\beta) < 0$. If $\beta < \beta_{\max}$, then $J_\beta(x_{\max}) = W(x_{\max})(J(x_{\max}) - \beta) = W(x_{\max})(\beta_{\max} - \beta) > 0$, so $j(\beta) = J_\beta(x_\beta) \geq J_\beta(x_{\max}) > 0$. For $\beta = \beta_{\max}$ we have

$$J_{\beta_{\max}}(x) = W(x)(J(x) - \beta_{\max}) \leq 0,$$

for all x and

$$J_{\beta_{\max}}(x_{\max}) = W(x_{\max})(J(x_{\max}) - \beta_{\max}) = 0,$$

so $j(\beta_{\max}) = \max_x J_{\beta_{\max}}(x) = 0$ and we can take $x_{\beta_{\max}} = x_{\max}$. The last statement of the theorem is the implication of the continuity of the functional J_β on the compact $[\beta_1, \beta_2] \times K$ \blacktriangle

Theorem 2. Let $W(x) \neq 0$ and $W_0(x)$ be functionals on a set K of a metric space

$$\begin{aligned} J(x) &\equiv \frac{W_0(x)}{W(x)}, \\ J_\beta(x) &\equiv W(x)(W_0(x) - \beta W(x)). \end{aligned}$$

Suppose that for every β the extremal problems

$$J \rightarrow \max_K, \quad J_\beta \rightarrow \max_K$$

have their solutions x_{\max}, x_{β} . Let

$$\beta_{\max} \equiv J(x_{\max}), \quad j(\beta) \equiv J_{\beta}(x_{\beta}).$$

Then

$$\begin{cases} j(\beta) < 0, & \beta > \beta_{\max}, \\ j(\beta) > 0, & \beta < \beta_{\max}, \\ j(\beta) = 0, & \beta = \beta_{\max}, \end{cases}$$

so β_{\max} is the only solution of the equation

$$j(\beta) = 0.$$

Functionals $J(x)$ and $J_{\beta_{\max}}(x)$ take their maxima at the same point $x_{\max} = x_{\beta_{\max}}$.

Prove. To prove the theorem we multiply the numerator and denominator of the fraction J by $W(x)$ and repeat the prove of the previous theorem

▲

The corollary of the theorems is the following procedure of solution of the problem (1). On the first step we solve the problem (2) for any arbitrary β ; then we calculate the function $j(\beta)$ and solve the scalar equation (4).

By calculation of $j(\beta)$ for any given β we can see the direction where root of the equation is situated. Then any iteration scheme can be applied to find the root with the necessary accuracy.

2 Examples

Let us consider some problems where the above algorithm can be applied.

Problem 1. Let $x \in [x_1, x_2]$, $W(x) = ax + b$, $W_0(x) = a_0x + b_0$, $ax_1 + b > 0$, $ax_2 + b > 0$, $a > 0$.

In this case

$$\begin{aligned} J(x) &\equiv \frac{W_0(x)}{W(x)} = \frac{a_0x + b_0}{ax + b}, \\ J_{\beta}(x) &\equiv W_0(x) - \beta W(x) = (a_0 - \beta a)x + b_0 - \beta b. \end{aligned}$$

It is clear that the solution of the problem (2) is

$$x_{\beta} = \begin{cases} x_1, & \beta \geq a_0/a, \\ x_2, & \beta < a_0/a, \end{cases}$$

so

$$j(\beta) = J_\beta(x_\beta) = \begin{cases} a_0x_1 + b_0 - \beta(ax_1 + b), & \beta \geq a_0/a, \\ a_0x_2 + b_0 - \beta(ax_2 + b), & \beta < a_0/a. \end{cases}$$

Note that

$$j\left(\frac{a_0}{a}\right) = b_0 - \frac{a_0b}{a} = \frac{ab_0 - ba_0}{a}.$$

The solution of the equation $j(\beta) = 0$ is

$$\beta_{\max} = J(x_{\max}) = \begin{cases} J(x_1), & j\left(\frac{a_0}{a}\right) > 0 \iff ab_0 - ba_0 > 0, \\ J(x_2), & j\left(\frac{a_0}{a}\right) \leq 0 \iff ab_0 - ba_0 \leq 0. \end{cases}$$

Hence, the solution of the problem (1)

$$x_{\max} = \begin{cases} x_1, & ab_0 - ba_0 > 0, \\ x_2, & ab_0 - ba_0 \leq 0. \end{cases}$$

We can also obtain the same result using the derivation

$$J'(x) = \frac{a_0b - ab_0}{(ax + b)^2}.$$

Problem 2. Let $x \in [x_1, x_2]$, $W(x) = ax^2 + bx + c$, $W_0(x) = a_0x^2 + b_0x + c_0$, $ax_1^2 + bx_1 + c > 0$, $ax_2^2 + bx_2 + c > 0$, $a > 0$.

We have now

$$\begin{aligned} J(x) &\equiv \frac{W_0(x)}{W(x)} = \frac{a_0x^2 + b_0x + c_0}{ax^2 + bx + c}, \\ J_\beta(x) &\equiv W_0(x) - \beta W(x) = (a_0 - \beta a)x^2 + (b_0 - \beta b)x + c_0 - \beta c. \end{aligned}$$

The solution of the problem $J_\beta(x) \rightarrow \max$ is one of the following three values x_1, x_2 and

$$x_3 = \frac{1}{2}(b_0 - \beta b) / (a_0 - \beta a)$$

(in the case when the last belongs to $[x_1, x_2]$; otherwise the solution is one of two values x_1, x_2). Hence,

$$j(\beta) = \begin{cases} \max\{J_\beta(x_1), J_\beta(x_2), J_\beta(x_3)\}, & x_3 \in [x_1, x_2], \\ \max\{J_\beta(x_1), J_\beta(x_2)\}, & x_3 \notin [x_1, x_2]. \end{cases}$$

Problem 3. Let $x \in [x_1, x_2]$, $f_0(x) > 0$, $f(x) > 1$,

$$J(x) = \frac{\ln f_0}{\ln f} \rightarrow \max.$$

The problem

$$\ln f_0 - \beta \ln f \rightarrow \max$$

is equivalent to

$$J_\beta(x) = \frac{f_0}{f^\beta} \rightarrow \max.$$

For this problem we construct the auxiliary problem

$$J_{\gamma,\beta} = f_0 - \gamma f^\beta \rightarrow \max.$$

Let $x_{\gamma,\beta}$ be a solution of the last problem, $\gamma = \gamma(\beta)$ a solution of the equation

$$J_{\gamma,\beta}(x_{\gamma,\beta}) = 0$$

for a fixed β . Next let β_{\max} be a solution of the equation

$$J_\beta(x_{\gamma(\beta),\beta}) = 1.$$

Then $x_{\gamma(\beta_{\max}),\beta_{\max}}$ is a solution of the initial problem.

Problem 4. Let $x \in K$, where $K = \{x : \|x\| \leq r\}$ is a solid sphere of a Hilbert space H , $W(x) = \langle w, x \rangle + h$, $h > r \|w\|$, $W_0 = \langle w_0, x \rangle + h_0$,

$$J(x) \equiv \frac{W_0(x)}{W(x)} = \frac{\langle w_0, x \rangle + h_0}{\langle w, x \rangle + h},$$

$$J_\beta(x) \equiv W_0(x) - \beta W(x) = \langle w_\beta, x \rangle + h_\beta,$$

$$w_\beta \equiv w_0 - \beta w, \quad h_\beta = h_0 - \beta h.$$

It is clear that the solution of the problem (2) has now the form

$$x_\beta = r \frac{w_\beta}{\|w_\beta\|}, \tag{5}$$

therefore,

$$j(\beta) = r \|w_0 - \beta w\| + h_0 - \beta h. \tag{6}$$

So, the problem of maximization of J has been transformed to the solution of the nonlinear equation (in unknown value β)

$$r \|w_0 - \beta w\| + h_0 - \beta h = 0. \quad (7)$$

Let us show that the curves $y = r \|w_0 - \beta w\|$ and $y = \beta h - h_0$ intersect. We have

$$\begin{aligned} \lim_{\beta \rightarrow \infty} \frac{r \|w_0 - \beta w\|}{\beta} &= r \|w\| \operatorname{sgn} \beta, \\ \lim_{\beta \rightarrow \infty} [r \|w_0 - \beta w\| - r \beta \operatorname{sgn} \beta \|w\|] &= \\ &= \lim_{\beta \rightarrow \infty} r \frac{\|w_0 - \beta w\|^2 - \|\beta w\|^2}{\|w_0 - \beta w\| + \|\beta w\|} = \\ &= r \lim_{\beta \rightarrow \infty} \frac{\langle w_0 - \beta w, w_0 - \beta w \rangle - \langle \beta w, \beta w \rangle}{\|w_0 - \beta w\| + \|\beta w\|} = \\ &= r \lim_{\beta \rightarrow \infty} \frac{\|w_0\|^2 - \beta \langle w_0, w \rangle - \beta \langle w, w_0 \rangle}{\|w_0 - \beta w\| + \|\beta w\|} = \\ &= -r \operatorname{sgn} \beta \operatorname{Re} \langle w_0, \tilde{w} \rangle, \quad \tilde{w} \equiv w / \|w\|. \end{aligned}$$

So,

$$r \|w_0 - \beta w\| = r \|w\| |\beta| - r \operatorname{sgn} \beta \operatorname{Re} \langle w_0, \tilde{w} \rangle + o(1), \quad \beta \rightarrow \infty. \quad (8)$$

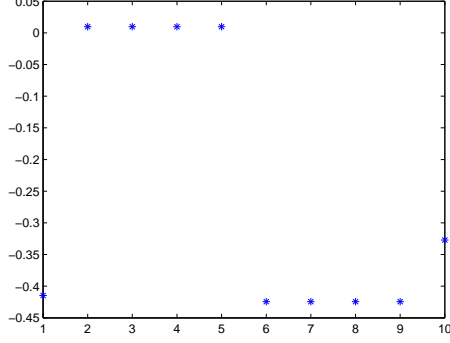
Under the condition $h_0 + r \|w_0\| > 0$ the curves intersect for positive β . Under the condition $h_0 + r \|w_0\| \leq 0$ they intersect for $\beta \leq 0$. So, $\beta_{\max} > 0$ in the first case and $\beta_{\max} \leq 0$ in the second. The cause of it is that the functional J takes some positive values in the first case and only non positive in the second.

Substituting the asymptotes (8) to the equation (7), we get a priori valuations of maximal value of J :

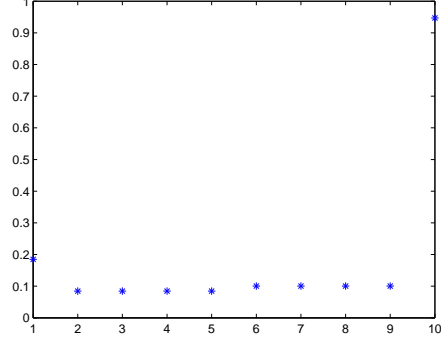
$$\beta_{\max} = \max J \simeq \begin{cases} \frac{h_0 - r \operatorname{Re} \langle w_0, \tilde{w} \rangle}{h - r \operatorname{Re} \langle w, \tilde{w} \rangle}, & h_0 + r \|w_0\| > 0, \\ \frac{h_0 + r \operatorname{Re} \langle w_0, \tilde{w} \rangle}{h + r \operatorname{Re} \langle w, \tilde{w} \rangle}, & h_0 + r \|w_0\| \leq 0. \end{cases} \quad (9)$$

Note that this valuations are correct only for big values of $|\beta_{\max}|$.

Example 1. Let $H = \mathbf{R}^{10}$, $w_0 = (1, 1, 1, 1, 1, 0, 0, 0, 0, 10)$,



(a) Example 1



(b) Example 2

Figure 1: Optimal vectors $x_{\max} = x_{\beta_{\max}}$

$w = (1, 0, 0, 0, 0, 1, 1, 1, 1, 1)$, $r = 1$, $h_0 = 15$, $h = 2.7$.

In this case $h_0 + r \|w_0\| \simeq 25.25 > 0$. The solution of the problem is presented on fig. 1a. The value $\max J \simeq 43.61$. Graphs of the functions $\beta \rightarrow j(\beta)$ and $\beta \rightarrow J(x_\beta)$ in $[0, \beta_{\max}]$ are presented on fig. 2a. The process of asymptotic estimations of the $\max J$ is illustrated by fig. 3a, the formula (9) for $\max J$ gives

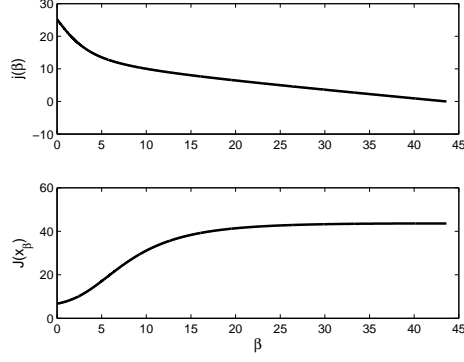
$$\max J \simeq \frac{h_0 - r \operatorname{Re} \langle w_0, \tilde{w} \rangle}{h - r \operatorname{Re} \langle w, \tilde{w} \rangle} \simeq 41.95.$$

Example 2. Let $H = \mathbf{R}^{10}$, $w_0 = (1, 1, 1, 1, 1, 0, 0, 0, 0, 10)$,

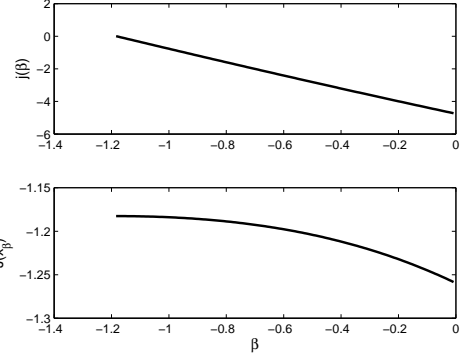
$w = (1, 0, 0, 0, 0, 1, 1, 1, 1, 1)$, $r = 1$, $h_0 = -15$, $h = 2.7$.

In this case $h_0 + r \|w_0\| \simeq -4.75 < 0$. The solution of the problem is presented on fig. 1b. The value $\max J \simeq -1.18$. Graphs of the functions $\beta \rightarrow j(\beta)$ and $\beta \rightarrow J(x_\beta)$ in $[0, \beta_{\max}]$ are presented on fig. 2b. The process of asymptotic estimations of the $\max J$ is illustrated by fig. 3b, the formula (9) for $\max J$ gives

$$\max J \simeq \frac{h_0 + r \operatorname{Re} \langle w_0, \tilde{w} \rangle}{h + r \operatorname{Re} \langle w, \tilde{w} \rangle} \simeq -2.04.$$

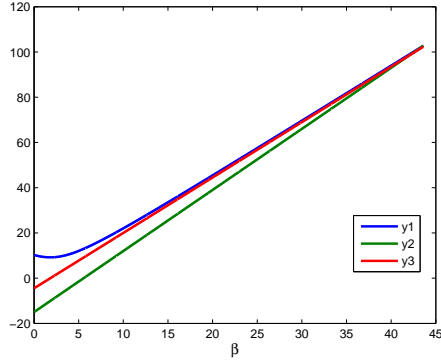


(a) Example 1

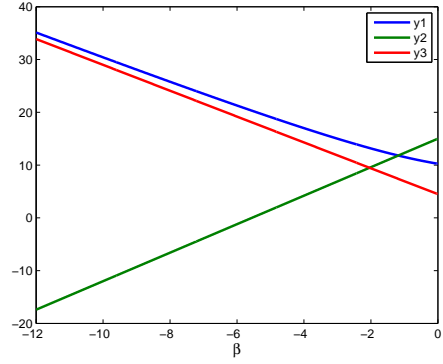


(b) Example 2

Figure 2: Graphs of functions $\beta \rightarrow j(\beta)$ (upper panels) and functions $\beta \rightarrow J(x_\beta)$ (lower panels).



(a) Example 1



(b) Example 2

Figure 3: Graphs of the functions $y_1(\beta) = r \|w_0 - \beta w\|$, $y_2(\beta) = \beta h - h_0$ and $y_3(\beta) = -r \|w\| \beta + r \operatorname{Re} \langle w_0, \tilde{w} \rangle$.